ON THE GENUS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. We define the class of Left Located Divisor (LLD) meromorphic functions and their vertical order $m_0(f)$ and their convergence exponent $d(f)$. When $m_0(f) \leq d(f)$ we prove that their Weierstrass genus is minimal. This explains the phenomena that many classical functions have minimal Weierstrass genus, for example Dirichlet series, the Γ-function, and trigonometric functions.

1. LLD meromorphic functions. Meromorphic functions f on $\mathbb{C}$, of the variable $s \in \mathbb{C}$, considered in this article are assumed to be of finite order $o = o(f)$. We recall that the order $o(f)$ is defined as

$$o(f) = \limsup_{R \to +\infty} \frac{\log \log ||f||_{C^0(B(0,R))}}{\log R} .$$

We study in this article Dirichlet series, and more generally the class of meromorphic of finite order with Left Located Divisor (LLD), which we call LLD meromorphic functions:

Definition 1. ***(LLD meromorphic functions)*** *A LLD meromorphic function is a function f of finite order and left located divisor*

$$\sigma_1 = \sup_{\rho \in f^{-1}(\{0,\infty\})} \Re \rho < +\infty .$$

The properties that we establish in this article are invariant by a real translation. Thus considering $g(s) = f(s + \sigma_1)$ instead of f we will assume that $\sigma_1 = 0$.

Examples of LLD meromorphic functions are Dirichlet series, that we normalize in this article such that $f(s) \to 1$ when $\Re s \to +\infty$. A Dirichlet series is of the form

$$f(s) = 1 + \sum_{n \geq 1} a_n \, e^{-\lambda_n s} , \tag{1}$$

with $a_n \in \mathbb{C}$ and

$$0 < \lambda_1 < \lambda_2 < \dots$$

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with (λ_n) a discrete set, that is either finite or $\lambda_n \to +\infty$, and such that we have a half plane of absolute convergence, i.e., for some $\bar{\sigma} \in \mathbb{R}$ we have

$$\sum_{n \geq 1} |a_n| \; e^{-\lambda_n \bar{\sigma}} < +\infty \, .$$

We refer to [6] for classical background on Dirichlet series.

2. **Convergence exponent.** We denote by (ρ) the set of zeros and poles of f, and the integer n_ρ is the multiplicity of ρ (positive for zeros and negative for poles, with the convention $n_\rho = 0$ if ρ is neither a zero nor a pole).

Definition 2. ***(Convergence exponent)*** *The convergence exponent of f is the minimum integer $d = d(f) \geq 0$ such that*

$$\sum_{\rho \neq 0} |n_\rho| \, |\rho|^{-d} < +\infty \ .$$

We have $d = 0$ if and only if f has a finite divisor, i.e. it is a rational function multiplied by the exponential of a polynomial, otherwise $d \geq 1$.

It is classical that the convergence exponent satisfies $d \leq [o] + 1$ (see [1]), thus it is finite for functions of finite order, but there is no upper bound of the order by the convergence exponent since we can always multiply by $\exp P$, where P is a polynomial, increasing the order without changing the divisor, hence keeping the same convergence exponent.

3. **Genus.** When f is a meromorphic function of finite order we have the Hadamard factorization of f (see [1], p.208)

$$f(s) = s^{n_0} e^{Q_f(s)} \prod_{\rho \neq 0} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \ ,$$

where

$$E_n(z) = (1-z) e^{z + \frac{1}{2} z^2 + \ldots + \frac{1}{n} z^n} \ ,$$

and Q_f is a polynomial, the Weierstrass polynomial, uniquely defined up to the addition of an integer multiple of $2\pi i$.

The *discrepancy polynomial* of the meromorphic function f is

$$P_f = -Q_f' \ .$$

We define the Hadamard part of f as

$$f_H(s) = s^{n_0} \prod_{\rho \neq 0} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \ . \tag{2}$$

Note that $f'/f = f_H'/f_H - P_f$.

The degree $g_W = \deg Q_f$ is the Weierstrass genus. The genus of f is defined as the integer

$$g = g(f) = \max(g_W(f), g_H(f)) \ .$$

where $g_H(f) = d(f)-1$ is the Hadamard genus, which is the degree of the polynomials in the exponential of the factors $E_n(z)$. From the definition we have $d \leq g+1$, and $g \leq o \leq g+1$ (see [1], p.209).

We set the following useful definition:

Definition 3. ***(Hadamard and Weierstrass type)*** *A meromorphic function f is of Hadamard type when $g(f) = g_H(f) = d(f) - 1 \geq g_W(f)$. It is of Weierstrass type when $g(f) = g_W(f) > g_H(f)$.*

Many classical functions are of Hadamard type. One of the purposes of the article is to explain why this holds.

4. **Vertical order.** For a LLD meromorphic function we look at the growth of its logarithmic derivative on the right half plane. This growth is always polynomial (proof in Appendix 1).

Proposition 4. *The logarithmic derivative of a LLD meromorphic function has polynomial growth on a right half plane, i.e. for $\sigma_2 > \max(0, \sigma_1)$, and for $\Re s > \sigma_2$,*

$$\left|\frac{f'(s)}{f(s)}\right| \leq C_0 |s|^{\max(d, g_W - 1)} \ ,$$

more precisely we have

$$\left|\frac{f_H'(s)}{f_H(s)}\right| \leq C_0 |s|^d$$

Remark 5. The exponent d is best possible in the last estimate (see the example constructed in Appendix 2).

We define the *vertical order* as follows:

Definition 6. ***(Vertical order)*** *The vertical order of a meromorphic function f with left located divisor is the minimal integer $m_0 = m_0(f) \geq 0$ such that for $c > \sigma_1$, $c \neq 0$,*

$$|c+it|^{-m_0} \frac{f'}{f}(c+it) \in L^1(\mathbb{R}) \ .$$

Lemma 7. *This definition does not depend on the choice of c.*

This Lemma is proved in Appendix 3.

From the estimate in Proposition 4 we have that $m_0(f_H) \leq d+2$. But we can do better:

Proposition 8. *We have $m_0(f_H) \leq d+1$.*

For a Dirichlet series normalized as in (1) we have that $f(s) \to 1$ and $f'(s) \sim -\lambda_1 a_1 e^{-\lambda_1 s}$ uniformly with $\Re s \to +\infty$, thus $m_0(f) = 2$.

In this article we say that a distribution has order n if n is the minimal integer such that it is the n-th derivative of a continuous function (there is no consensus in the classical literature on the definition of order of a distribution, for example see [8] and [10]). Proposition 4 implies that the inverse Laplace transform $\mathcal{L}^{-1}(f'/f)$ is a distribution of finite order. This is because we have an explicit formula for the inverse Laplace transform. We recall (see [10]) that

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(c+iu) e^{(c+iu)t} \, du \ ,$$

if the integral is convergent, and

$$\mathcal{L}^{-1}(F)(t) = \mathcal{L}_c^{-1}(F)(t) = \frac{1}{2\pi} \frac{D^n}{Dt^n} \int_{\mathbb{R}} \frac{F(c+iu)}{(c+iu)^n} e^{(c+iu)t} \, du \ ,$$

in general (the derivative is taken in distributional sense) which holds for some n when F is holomorphic with polynomial growth on $\{\Re s > \sigma_2\}$ and it is independent of $c > \sigma_2 > \sigma_1$.

A closely related integer to the vertical order is the *distributional vertical order.*

Definition 9. ***(Distributional vertical order)*** *The distributional vertical order of a LLD meromorphic function f is the minimal integer $m \geq 0$ such that the inverse Laplace transform*

$$\mathcal{L}^{-1}(f'/f)$$

is a distribution of order m.

It is clear that:

Proposition 10. *We have $m(f) \leq m_0(f)$.*

5. Main results.

Theorem 11. *For a LLD meromorphic function f we have that if $m(f) \neq g_W(f)+1$ then f is of Hadamard type, i.e. $g_W(f) \leq g_H(f) = g(f)$.*

Moreover, any Dirichlet series f is of Hadamard type, i.e. $g_W(f) \leq g_H(f) = g(f)$ unconditionally.

Corollary 12. *If a LLD meromorphic function f is of Weierstrass type then $m(f) = g_W(f) + 1$.*

The next corollary gives an analytic criterium to determine if a meromorphic function is of Hadamard type.

Corollary 13. *If $m_0(f) \leq d(f)$ then f is of Hadamard type.*

The same argument used in the proof of the main theorem gives:

Theorem 14. *Let f be a non-constant Dirichlet series. Then we have*

$$d(f) \geq 2 \ ,$$

and

$$o(f) \geq 1 \ .$$

Before proving these results we need to introduce the Newton-Cramer distribution and Poisson-Newton formula.

6. **Newton-Cramer Distribution.** In [7] we associate to the divisor $\mathrm{div}(f) = \sum n_\rho \, \rho$ its Newton-Cramer distribution, which is given by the series

$$W(f) = \sum n_\rho \, e^{\rho t}$$

on $\mathbb{R}_+^*$. This sum is only converging in $\mathbb{R}_+^*$ in the distribution sense. The distribution $W(f)$ vanishes in $\mathbb{R}_-^*$, and has some structure at 0. The precise definition follows (we assume, in order to simplify, that $\rho = 0$ is not part of the divisor).

Definition 15. ***(Newton-Cramer distribution)*** *The Newton-Cramer distribution is*

$$W(f) = \frac{D^d}{Dt^d} \left(L_d(t) \right) ,$$

where L_d is the continuous function on $\mathbb{R}$ defined on $\mathbb{R}_+$ by

$$L_d(t) = \sum_{\rho \neq 0} \frac{n_\rho}{\rho^d} (e^{\rho t} - 1) \, \boldsymbol{1}_{\mathbb{R}_+} \ .$$

It is easy to see that the sum converges for $t \geq 0$.

In this article, only the order of distributions plays a role, and the space of test funcitons for which the distribution belong to the dual is not so important. The distribution $W(f)$ is Laplace transformable, that is, it can be paired with e^{-st} on $\mathbb{R}_+$, on some half-plane $\Re s > \sigma_0$. Hence, the appropriate space of distributions to use is the dual of the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}$ which decay faster than $Ce^{\alpha |t|}$, for some $C > 0$, $\alpha > 0$.

The main property of the Newton-Cramer distribution that we need follows from its definition:

Proposition 16. *The Newton-Cramer distribution is the d-th derivative of a continuous function.*

7. **Poisson-Newton formula.** The Newton-Cramer distribution of f is linked to the inverse Laplace transform of the logarithmic derivative f'/f by the Poisson-Newton formula (see [7]):

Theorem 17. ***(Poisson-Newton formula)*** *For a LLD meromorphic function f we have on $\mathbb{R}$*

$$W(f) = \sum_{l=0}^{g_W-1} c_l \delta_0^{(l)} + \mathcal{L}^{-1}(f'/f) \ ,$$

where $P_f(s) = c_0 + c_1 s + \ldots + c_{g_W-1} s^{g_W-1} = -Q'_f(s)$ is the discrepancy polynomial.

When f is a Dirichlet function, the Laplace transform $\mathcal{L}^{-1}(f'/f)$ is purely atomic with atoms in $\mathbb{R}_+^*$. We can compute it explicitely as follows. On the half plane $\Re s > \sigma_1$, $\log f(s)$ is well defined taking the principal branch of the logarithm. Then we can define the coefficients $(b_{\mathbf{k}})$ by

$$- \log f(s) = - \log \left(1 + \sum_{n \geq 1} a_n \, e^{-\lambda_n s} \right) = \sum_{\mathbf{k} \in \Lambda} b_{\mathbf{k}} \, e^{-\langle \boldsymbol{\lambda}, \mathbf{k} \rangle s} \ , \tag{3}$$

where $\Lambda = \{\mathbf{k} = (k_n)_{n\geq 1} \,|\, k_n \in \mathbb{N}, ||\mathbf{k}|| = \sum |k_n| < \infty, ||\mathbf{k}|| \geq 1\}$, and $\langle \boldsymbol{\lambda}, \mathbf{k} \rangle = \lambda_1 k_1 + \ldots + \lambda_l k_l$, where $k_n = 0$ for $n > l$. Note that the coefficients $(b_{\mathbf{k}})$ are polynomials on the (a_n). More precisely, we have

$$b_{\mathbf{k}} = \frac{(-1)^{||\mathbf{k}||}}{||\mathbf{k}||} \frac{||\mathbf{k}||!}{\prod_j k_j!} \prod_j a_j^{k_j} \ . \tag{4}$$

Note that if the λ_n are $\mathbb{Q}$-dependent then there are repetitions in the exponents of (3).

Since $\mathcal{L}(e^{-\lambda s}) = \delta_\lambda$, we have

$$\mathcal{L}^{-1}(f'/f) = \sum_{\mathbf{k} \in \Lambda} \langle \lambda, \mathbf{k} \rangle \, b_{\mathbf{k}} \ \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \ .$$

Note in particular that $\operatorname{supp} \mathcal{L}^{-1}(f'/f) \subset [\epsilon, +\infty[$ for some $\epsilon > 0$.

8. **Proof of the main results.** The proof of Theorem 11 consists on inspecting the orders of the distributions in both sides of the Poisson-Newton equation:

$$W(f) = \sum_{l=0}^{g_W-1} c_l \delta_0^{(l)} + \mathcal{L}^{-1}(f'/f) \ .$$

We will use that for two distributions U and V, if $\mathrm{ord}(U) \neq \mathrm{ord}(V)$ then

$$\mathrm{ord}(U+V) = \max(\mathrm{ord}(U), \mathrm{ord}(V)) \ .$$

The left hand side is of order $\leq d$ since $W(f)$ is the d-th derivative of a continuous function.

Observe that the Dirac δ_0 is of order 2, and $\delta_0^{(l)}$ is of order $l+2$. In particular, the first term of the right hand side in Poisson-Newton equation is of order $g_W + 1$.

The second term of the right hand side is of order $m(f)$ by definition of $m(f)$.

To prove Theorem 11 we assume first that $m < g_W + 1$. Then the order of the right hand side in Poisson-Newton formula is $g_W + 1$. Therefore $d \geq g_W + 1$ so $g = g_H \geq g_W$ and f is of Hadamard type.

We look at the second case when $m > g_W + 1$. Then the order of the right hand side is m, thus comparing with the left hand side, we get $d \geq m > g_W + 1$, therefore $g = g_H > g_W$ and f is again of Hadamard type. This proves the first statement of the main theorem.

For a Dirichlet series f the distribution $\mathcal{L}^{-1}(f'/f)$ has support away from 0, therefore looking at the local order at 0 (which is smaller or equal than the global order) of both sides of the equation we get that $d \geq g_W + 1$ unconditionally. This gives $g = g_H \geq g_W$ and f is always of Hadamard type. This ends the proof of Theorem 11.

Now Corollary 12 is a direct application of the main theorem.

For Corollary 13 we observe that $m_0(f) \leq d(f)$ gives $m(f) \leq m_0(f) \leq d(f) \leq g(f) + 1$. If the last inequality is an equality, then f is of Hadamard type and we are done. Otherwise we have $g = g_W$ and $m(f) < g_W(f) + 1$ and using the main theorem we get also that f is of Hadamard type, and $g = g_W = g_H = d - 1$.

For the proof of Theorem 14, we inspect as before the order of the distributions in the Poisson-Newton-formula. The right hand side contains Dirac distributions at the frequencies, hence it is at least a second derivative of a continuous function. In the left hand side we have $W(f)$ that is the d-th derivative of a continuous function. This gives $d \geq 2$.

Also we know that $d \leq o + 1$, hence $o \geq 1$.

9. **Proof of Poisson-Newton formula.** Let us prove Theorem 17. We start from the Hadamard factorization of f (assuming that $\rho = 0$ is not part of the divisor in order to simplify).

$$f(s) = e^{Q_f(s)} \prod_{\rho} \left(E_{d-1}(s/\rho)\right)^{n_\rho} \ ,$$

We take its logarithmic derivative:

$$
\begin{aligned}
f'/f &= -P_f + \sum_\rho n_\rho \frac{E'_{d-1}(s/\rho)}{E_{d-1}(s/\rho)} \\
&= -P_f + \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \qquad (5)
\end{aligned}
$$

Since for $l \geq 0$

$$
\mathcal{L}(\delta_0^{(l)}) = s^l \ ,
$$

the polynomial P_f is the Laplace transform

$$
P_f = \mathcal{L}\left(c_0\delta_0 + c_1\delta'_0 + \ldots + c_{g-1}\delta_0^{(g-1)} \right) \ .
$$

It remains to prove that

$$
\mathcal{L}(W(f)) = \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \ .
$$

We have

$$
\frac{D^d}{Dt^d}\left((e^{\rho t} - 1)\mathbf{1}_{\mathbb{R}_+} \right) = \rho^d e^{\rho t}\mathbf{1}_{\mathbb{R}_+} + \sum_{l=0}^{d} \rho^{d-1-l}\delta_0^{(l)} \ ,
$$

thus for a finite set A of zeros and poles of the divisor, we have

$$
\begin{aligned}
W_A(f) &= \sum_{\rho\in A} n_\rho \rho^{-d} \frac{D^d}{Dt^d}\left(e^{\rho t} - 1 \right) \mathbf{1}_{\mathbb{R}_+} \\
&= \sum_{\rho\in A} n_\rho \left(e^{\rho t}\mathbf{1}_{\mathbb{R}_+} + \sum_{l=0}^{d} \rho^{-1-l}\delta_0^{(l)} \right) \ .
\end{aligned}
$$

Now we have

$$
\mathcal{L}(e^{\rho t}\mathbf{1}_{\mathbb{R}_+}) = \frac{1}{\rho - s} \ ,
$$

so

$$
\mathcal{L}\left(W_A(f)\right) = \sum_{\rho\in A} n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{s^l}{\rho^{l+1}} \right) \ ,
$$

and we are done taking the inverse Laplace transform.

10. **Application to trigonometric functions.** We check that the sine function is of Hadamard type. For this it is enough to consider the hyperbolic sine function which is an entire function of order 1,

$$f(s) = \sinh(s) = \frac{e^s - e^{-s}}{2i} \ .$$

The zeros are, for $k \in \mathbb{Z}$,

$$\rho_k = \pi i k \ ,$$

thus sinh is a LLD entire function and $d(f) = 2$. Also we have

$$f'(s)/f(s) = \cosh(s)/\sinh(s) = \frac{1 + e^{-2s}}{1 - e^{-2s}} \to 1$$

when $\Re s \to +\infty$. Therefore $m_0(f) = 2$.

Using Corollary 13 we get

Proposition 18. *The function $f(s) = \sinh(s)$ is of Hadamard type.*

This is something that we know from its Hadamard factorisation (due to Euler)

$$\sinh(s) = s \prod_{k \in \mathbb{Z}^*} \left(1 - \frac{s}{\pi i k}\right) e^{\frac{s}{\pi i k}} \ .$$

Corollary 19. *The function $f(s) = \sin(s)$ is of Hadamard type.*

11. **Application to the Γ function.** We check, without computing its Hadamard factorisation, that the classical Γ function is of Hadamard type.

The Γ-function has no zeros and has simple poles at the negative integers. Thus it is a LLD meromorphic function and $d = 2$. Stirling formula indicates that we must have $m_0(\Gamma) = 2$ and we check:

Lemma 20. *For $c > 0$ we have for some constant $C_0 > 0$ and for $|u| \geq 1$*

$$\left| \frac{\Gamma'}{\Gamma}(c + iu) \right| \leq \log |u| + C_0$$

and $m_0(\Gamma) = 2$.

The classical Stirling's asymptotics holds in a right cone, but we need the estimate in a vertical line, thus we need to refine the classical estimate. We start with Binet's second formula (see [9] p.251):

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + \varphi(s) \ ,$$

where

$$\varphi(s) = 2 \int_0^{+\infty} \frac{\arctan(t/s)}{e^{2\pi t} - 1}\, dt \ .$$

Taking one derivative in the above formula, we get an identity for the digamma function

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \varphi'(s) \ ,$$

and

$$\varphi'(s) = -2 \int_0^{+\infty} \left(\frac{s}{s^2+t^2} \right) \left(\frac{t}{e^{2\pi t} - 1} \right) dt \ .$$

Since

$$\int_0^{+\infty} \frac{t}{e^{2\pi t} - 1}\, dt = \frac{B_2}{4} = \frac{1}{24} \ ,$$

and if $s = c + iu$ with $c = \Re s > 0$,

$$\left| \frac{s}{s^2+t^2} \right| \leq \frac{1}{|c|} \ ,$$

we have the estimate

$$|\varphi'(s)| \leq \frac{1}{24|c|} \ ,$$

so $|\psi(s)| \leq \log |s| + C_0$, and the lemma follows.

Now we have $m_0(\Gamma) = 2 \leq d(\Gamma) = 2$ so the application of Corollary 13 gives:

Proposition 21. *The Γ function is a meromorphic function of Hadamard type.*

12. Application to the Riemann zeta function. The Riemann zeta function is a Dirichlet series,

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-s} \ ,$$

and has a meromorphic extension of order 1 to the whole complex plane. So it is a LLD meromorphic function.

We that $d(\zeta) \leq 2$ from the order 1, and $d(\zeta) \geq 2$ for the summation of the trivial zeros that lie at the even negative integers, thus $d(\zeta) = 2$.

The logarithmic derivative is bounded on vertical lines and so $m_0(\zeta) = 2$. Again, using Corollary 13 we get:

Proposition 22. *The Riemann zeta function ζ is a meromorphic function of Hadamard type.*

13. Appendix 1: Proof of propositions 4 and 8. We start by considering the analogue of (5) centered at σ_1. This is

$$f'/f = -P_f + \sum_\rho n_\rho \left(\frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{(s-\sigma_1)^l}{(\rho-\sigma_1)^{l+1}} \right)$$

We write $f'/f = -P_f + G$, where $G(s) = \sum n_\rho \, g_\rho(s)$, where

$$g_\rho(s) = \frac{1}{\rho - s} + \sum_{l=0}^{d-2} \frac{(s-\sigma_1)^l}{(\rho-\sigma_1)^{l+1}} = \frac{(s-\sigma_1)^{d-1}}{(\rho-\sigma_1)^{d-1}} \frac{1}{\rho - s}$$

In order to prove Proposition 4, we need to bound $|g_\rho(s)| \leq C|s-\sigma_1|^d|\rho-\sigma_1|^{-d}$, for a uniform constant C, since $\sum n_\rho \, |\rho-\sigma_1|^{-d} < \infty$. For this we need to bound uniformly

$$g_\rho(s) = \frac{\rho - \sigma_1}{(s-\sigma_1)(\rho - s)}$$

on the half-plane $\Re s > \sigma_2$.

If $|\sigma_1 - s| \leq \frac{1}{2}|\rho - \sigma_1|$ then $|\rho - s| \geq |\rho-\sigma_1| - |\sigma_1 - s| \geq \frac{1}{2}|\rho-\sigma_1|$. So $|g_\rho(s)| \leq \frac{2}{|s-\sigma_1|} \leq C$, as $|s-\sigma_1| \geq \sigma_2 - \sigma_1$ is bounded below.

If $|\sigma_1 - s| \geq \frac{1}{2}|\rho - \sigma_1|$ then $|g_\rho(s)| \leq \frac{2}{|\rho - s|} \leq C$, as $|s - \rho| \geq \sigma_2 - \sigma_1$ is bounded below.

We prove now Proposition 8. Fix $c > \sigma_1$, and let $a = c - \sigma_1 > 0$. We need to see that $G(s)|s-\sigma_1|^{-d-1}$ is integrable, and it is enough to see that

$$\int_{L_c} \frac{|\rho-\sigma_1|}{|s-\sigma_1|^2|\rho - s|} ds \tag{6}$$

is bounded uniformly on ρ, for $L_c = c + i\mathbb{R}$.

We consider two sets:

- $A = \{s \in L_c \,|\, |\rho - \sigma_1| \leq \frac{3}{2}|\rho - s|\}$. This is an infinite portion of L_c. The integral is bounded by
$$\frac{3}{2}\int_{L_c} \frac{1}{|s-\sigma_1|^2} ds < \infty.$$
- $B = \{s \in L_c \,|\, |\rho - \sigma_1| \geq \frac{3}{2}|\rho - s|\}$. This is the intersection of a disc of radius $\frac{2}{3}|\rho-\sigma_1|$ with L_c. So its length is bounded by $\frac{4}{3}|\rho-\sigma_1|$. The integral there is bounded by
$$\frac{4}{3}\max\left\{ \frac{|\rho-\sigma_1|^2}{|s-\sigma_1|^2|\rho - s|} \,|\, s \in B \right\}.$$

We have that $|\rho - s| \geq a$, so $|\rho - s|^{-1/2} \leq \frac{1}{\sqrt{a}} \leq \frac{1}{2}$, for $a \geq 4$. Then $|\rho - s| + |\rho - s|^{1/2} \leq \frac{3}{2}|\rho - s|$ and

$$|\rho - s| + |\rho - s|^{1/2} \leq |\rho - \sigma_1| \leq |\rho - s| + |s - \sigma_1|.$$

So $|\rho - s|^{1/2} \leq |s - \sigma_1|$ and

$$\frac{|\rho-\sigma_1|^2}{|s-\sigma_1|^2|\rho - s|} \leq \frac{(|\rho - s| + |s-\sigma_1|)^2}{|s-\sigma_1|^2|\rho - s|} \leq \frac{1}{|\rho - s|} + \frac{2}{|s-\sigma_1|} + \frac{|\rho - s|}{|s-\sigma_1|^2} \leq 1 + \frac{3}{a}\,.$$

This proves that (6) is uniformly bounded.

14. Appendix 2: The exponent d in Proposition 4 is best possible. We construct an example that has the sharp exponent.

We construct a meromorphic function with convergence exponent $d = 1$. More precisely, let f be an entire function with zeros at $\rho = n^2 2^n i$, $n \geq 1$, and with multiplicities $n_\rho = 2^n$. Then $\sum n_\rho |\rho|^{-1} < \infty$. The logarithmic derivative of such function is given by

$$g = \frac{f'}{f} = \sum \frac{n_\rho}{s - \rho}$$

Now let us see that it is not controlled as $|f'/f| \leq C|s|^{1-\epsilon}$ with $\epsilon > 0$. For this take $s = c + k^2 2^k i$, k a fixed integer, $c > 0$. We decompose

$$g(s) = \sum_{n=1}^{k-1} \frac{2^n}{c + (k^2 2^k - n^2 2^n)i} + \frac{2^k}{c} + \sum_{n=k+1}^{\infty} \frac{2^n}{c + (k^2 2^k - n^2 2^n)i}$$

The first term is bounded by

$$\sum_{n=1}^{k-1} \frac{2^n}{k^2 2^k - (k-1)^2 2^{k-1}} \leq \frac{2^{k-1}}{2^{k-1}(k^2 + 2k - 1)} < C_0,$$

for some universal constant. The third term is bounded by

$$\sum \frac{2^n}{n^2 2^n - k^2 2^k} \leq \sum \frac{2^n}{n^2 2^{n-1}} < C_1,$$

for another universal constant. Hence $|g(s)| \geq \frac{2^k}{c} - C_0 - C_1$. For fixed c, take k large enough. Then

$$\frac{|g(s)|}{|s|^{1-\epsilon}} \geq \frac{2^k/c - C_0 - C_1}{(c^2 + k^4 2^{k+1})^{(1-\epsilon)/2}} \approx \frac{2^\epsilon}{c\, k^{2-2\epsilon}},$$

which gets as large as we wish.

15. Appendix 3: Proof of Lemma 7. Fix $c > \sigma_1$ and let m_0 be the minimal integer such that

$$\left|(c+it)^{-m_0}\frac{f'}{f}(c+it)\right| \in L^1(\mathbb{R}) \ .$$

Consider the holomorphic function

$$g(s) = s^{-m_0}\frac{f'(s)}{f(s)}$$

on the right half-plane $\Re s \geq c$. The function $F(t) = g(c+it)$ satisfies the conditions of the Representation Theorem 6.5.4 in [2] with $\alpha = 0$, $c = 0$, and we get using the last inequality of that Theorem

$$\log|g(c'+iu)| \leq (c'-c)\pi^{-1}\int_{\mathbb{R}}\frac{\log|g(c+it)|}{(t-u)^2+(c'-c)^2}\ dt \ .$$

Now taking the exponential and using Jensen's convexity inequality we get

$$|g(c'+iu)| \leq (c'-c)\pi^{-1}\int_{\mathbb{R}}\frac{|g(c+it)|}{(t-u)^2+(c'-c)^2}\ dt \ .$$

Now Fubini gives

$$\int_{\mathbb{R}}|g(c'+iu)|du \leq (c'-c)\pi^{-1}\int_{\mathbb{R}}\left(|g(c+it)|\int_{\mathbb{R}}\frac{1}{(t-u)^2+(c'-c)^2}\ du\right)dt = \int_{\mathbb{R}}|g(c+it)|dt < \infty.$$

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